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CRITERION EQUATIONS OF THE HEAT AND MASS TRANSFER AT THE SURFACE OF DROPLETS AND VAPOUR BUBBLES

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Abstract—For the general description of the heat and mass transfer at the interface of two-phase media a system of similarity criteria was elaborated which consists of the Euler, Froude, Galilei, Jakob, Nusselt, Péclét, Reynolds, Stanton and Weber numbers adapted for droplets and vapour bubbles. In order to make possible taking into account the deviations caused by the dispersed state of the particles also the notion of capillary latent heat was defined.

NOMENCLATURE

A,	area of phase interface;
a,	thermal diffusivity, $= \lambda/(c_p\rho)$;
c_p,	isobaric specific heat;
E,	internal energy;
H,	enthalpy;
h,	specific enthalpy;
p,	pressure;
p_s,	saturation pressure;
q,	heat flux density;
R,	radius;
r,	latent heat of vaporization;
T,	temperature;
t,	time;
V,	volume;
α,	heat-transfer coefficient;
λ,	thermal conductivity;
ν,	kinematic viscosity;
ρ,	mass density;
σ,	surface tension;
$\mathrm{d}R/\mathrm{d}t$,	radial velocity;
$\mathrm{d}^2R/\mathrm{d}t^2$,	radial acceleration;
Δp,	pressure difference;
ΔT,	temperature difference;
$'$,	refers to liquid phase;
$''$,	refers to vapour phase;
$*$,	refers to disperse state.

THE BEHAVIOUR of droplets and bubbles has been repeatedly investigated regarding both theoretical and experimental aspects, since direct contact between vapour and liquid often occurs in industrial processes. Systematizing and generalizing experimental results can be carried out in the most appropriate way by means of similarity criteria.

The criterion equation of heat and mass transfer for droplets and bubbles will be deduced from Newton's basic equation of heat transfer per unit area

$$q = \alpha\Delta T \tag{1}$$

which is a relation of universal validity.

The special form of the basic equation (1) adequate for vapour bubbles in liquid was established by Bošnjakovič, based on the theory that the temperature difference which maintains the radial motion of the phase interface is localized in a thin thermal boundary layer surrounding the bubble [1–3]. For the bubble growth in superheated liquid the heat balance at the phase interface

$$\alpha\Delta T = r\rho\,\mathrm{d}R/\mathrm{d}t \tag{2}$$

was obtained.

The size variation during the growth and collapse of vapour bubbles may be described in the same way, since both of these processes are of identical nature but of reverse direction. Therefore the equation (2) is valid not only for the growth of bubbles in superheated liquid, but also for both their growth and collapse in subcooled liquid [2, 4].

It was established by Plesset and Zwick that the temperature and pressure in vapour bubble surrounded by superheated liquid is decreased necessarily when it is growing, because of the heat input and evaporation at the bubble boundary [5]. One has no reason to suppose that the analogue of this phenomenon does not proceed under adequate conditions when the bubble is collapsing. Consequently, it is to be expected that in case of bubble collapse in subcooled liquid the temperature and pressure within the bubble are increased owing to the heat output and condensation at the bubble boundary. This supposition is in keeping with the results yielded by Akiyama's theoretical and experimental investigation [6].

The heat input of evaporation and the heat output of condensation, however, depend on the rate of size variation, so the heat-transfer problem is coupled with mass transfer and dynamic problems.

It follows from the foregoing that in case of droplets the heat balance at the phase interface has the same form as the equation (2) which was originally defined for bubbles, but it is to be noted that the latent heat is of capillary character in both cases, consequently, it differs from the usual one, as will be seen in the sequel.

It is known that according to Laplace's theorem about capillary superpressure the pressure difference between the droplet and surrounding vapour, or between the bubble and surrounding liquid, respectively, is defined by

$$\Delta p = \frac{2\sigma}{R} \tag{3}$$

what was transformed by Kelvin, considering that the vapour pressure deviates from the saturation pressure corresponding to its own temperature, by

$$\Delta p'' = p_s - p'' = \mp \frac{2\sigma}{R} \frac{\rho''}{\rho' - \rho''}, \tag{4}$$

where the upper sign (−) relates to the droplet, and the lower one (+) to the bubble [3].

This means that the difference between saturation pressure and bubble pressure is immediately given by the equation (4), but for determining the difference between saturation pressure and droplet pressure the equation (4) should be completed by the pressure difference between phases Δp (3). Based on this assumption, the deviation of the pressure within disperse particles of both kinds from the saturation pressure can be defined by

$$\Delta p_* = \mp \frac{2\sigma}{R} \frac{\rho}{\rho' - \rho''}, \tag{5}$$

where the signs are to be used as above, and the density ρ in numerator always belongs to the disperse phase.

Since between the phases there is a temperature difference, the droplet or bubble, respectively, has a radially moving boundary, in consequence of what dynamic and thermal effects are produced which are not shown by the equations (3)–(5), therefore these relations may be regarded as estimations reflecting the primary effect of the surface energy.

Droplets and bubbles do not show necessarily uniform or monotonic size variation, the radial motion may be also of oscillatory character [6], in consequence of secondary effects superposed on the primary one. In spite of additional effects, it can undoubtedly be established that the driving force of the process in question is all the time the temperature difference between phases [2, 5, 7], on the one hand, and the interdependence of temperature difference and pressure difference is determined by the Clausius–Clapeyron equation [8–10], on the other hand.

Moreover, it is known that for the internal energy of the surrounded particle belonging to the disperse phase

$$E_* = E + \left(\sigma - T\frac{\mathrm{d}\sigma}{\mathrm{d}T}\right)A \tag{6}$$

is obtained, as deduced by Keenan [11], Grigull, Bach, and Straub [12–14], as well as, by Novikov and Voskresenskiy [15]. Consequently, the enthalpy of a single disperse particle can be defined by

$$H_* = E_* + p_* V, \tag{7}$$

where the pressure p_* is higher or lower by Δp_* (5) than the saturation pressure corresponding to its own temperature, what results in

$$H_* = H + \left(\sigma - T\frac{\mathrm{d}\sigma}{\mathrm{d}T}\right)A - \Delta p_* V \tag{8}$$

for the enthalpy of a single disperse particle, the specific enthalpy per unit mass of the disperse phase having uniform temperature and size distribution takes the shape

$$h_* = h + \frac{3}{R\rho}\left[\sigma\left(1 \pm \frac{2}{3}\frac{\rho}{\rho' - \rho''}\right) - T\frac{\mathrm{d}\sigma}{\mathrm{d}T}\right], \tag{9}$$

and, other things being equal, for the latent heat of vaporization

$$r_* = r \mp \frac{3}{R\rho}\left[\sigma\left(1 \pm \frac{2}{3}\frac{\rho}{\rho' - \rho''}\right) - T\frac{\mathrm{d}\sigma}{\mathrm{d}T}\right] \tag{10}$$

is yielded, in which the upper signs relate to the droplet, and the lower ones to the bubble.

As can be seen, a perceptible modification of the latent heat is caused by sizes of capillary magnitude, therefore the notion defined by the equation (10) may be regarded as *capillary latent heat*, opposite to the usual one which is of macroscopic character.

The phenomena in question can be described also by means of similarity criteria, as follows.

It was proposed by Ellion [16] and Zuber [2] that the *Reynolds number* for bubble be defined using the bubble radius for a characteristic length, and the radial velocity of bubble boundary for a significant one.

On this analogy it will be suggested to introduce further similarity criteria, modifying the usual ones with regard not only to bubbles but also to droplets, in a manner more appropriately than proposed earlier [17].

In case we apply the definitions

$$Re_* = \frac{R(\mathrm{d}R/\mathrm{d}t)}{\nu}, \tag{11}$$

$$Nu_* = \frac{\alpha R}{\lambda}, \tag{12}$$

$$Ja_* = \frac{\Delta T c'_p \rho'}{r_* \rho''}, \tag{13}$$

$$Pe_* = \frac{R(\mathrm{d}R/\mathrm{d}t)}{a}, \tag{14}$$

$$St_* = \frac{\alpha}{c_p \rho (\mathrm{d}R/\mathrm{d}t)}, \tag{15}$$

$$Pr = \frac{\nu}{a}, \tag{16}$$

where the last one is unchanged, an opportunity presents itself to express by means of them the radial velocity, the heat-transfer coefficient, and the temperature difference between phases in various ways. Substituting them into the equation (2), the heat balance turns to

$$\frac{Nu'_* Ja'_*}{Re'_* Pr'} = \frac{Nu'_* Ja'_*}{Pe'_*} = St'_* Ja'_* = \frac{\rho'}{\rho''} \tag{17'}$$

$$\frac{Nu''_* Ja''_*}{Re''_* Pr''} = \frac{Nu''_* Ja''_*}{Pe''_*} = St''_* Ja''_* = \frac{c'_p \rho'}{c''_p \rho''} \tag{17''}$$

representing nothing else but Bošnjakovič's conceptual model (2) in dimensionless formulation (17″) and its analogue for droplets (17′).

Similarly to the heat balance, the pressure difference (5), too, can be transformed into a criterion equation, as follows.

Applying for significant ones the radial acceleration of the phase interface instead of gravitation, on the one hand, and the pressure difference caused by the radial motion of the phase interface, on the other hand, we obtain the definitions

$$Eu_* = \frac{\Delta p_*}{\rho (\mathrm{d}R/\mathrm{d}t)^2}, \tag{18}$$

$$We_* = \frac{\sigma}{(\mathrm{d}^2R/\mathrm{d}t^2)(\rho' - \rho'')R^2}, \tag{19}$$

$$Ga_* = \frac{R^3(\mathrm{d}^2R/\mathrm{d}t^2)}{\nu^2}, \tag{20}$$

$$Fr_* = \frac{(\mathrm{d}R/\mathrm{d}t)^2}{R(\mathrm{d}^2R/\mathrm{d}t^2)}. \tag{21}$$

Having expressed the pressure difference and surface tension, as well as substituting them into the equation (5), we obtain the criterion equation of the difference between the pressure of single surrounded particles and saturation pressure corresponding to its own temperature, what is defined by

$$\frac{Eu_*(Re_*)^2}{We_* Ga_*} = \frac{Eu_* Fr_*}{We_*} = 2, \tag{22}$$

based on Laplace and Kelvin's theorem about the capillary superpressure.

As can be seen, the *Reynolds number* is contained in the equations (17′), (17″), and (22). Uniting one of the formers with the latter by the *Reynolds number*, an opportunity offers itself to produce the *Nusselt numbers*

$$Nu'_* = \frac{\rho'}{\rho''} \frac{Pr'}{Ja'_*} \sqrt{\frac{2We_* Ga'_*}{Eu'_*}} = \frac{\rho'}{\rho''} \frac{Pe'_*}{Ja'_*} \sqrt{\frac{2We_*}{Eu'_* Fr'_*}}, \tag{23'}$$

$$Nu''_* = \frac{c'_p \rho'}{c''_p \rho''} \frac{Pr''}{Ja''_*} \sqrt{\frac{2We_* Ga''_*}{Eu''_*}} = \frac{c'_p \rho'}{c''_p \rho''} \frac{Pe''_*}{Ja''_*} \sqrt{\frac{2We_*}{Eu''_* Fr''_*}}, \tag{23''}$$

what can be regarded for the criterion equations of heat and mass transfer in case of disperse, two-phase, one-component media.

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EQUATIONS CRITERES DU TRANSFERT DE CHALEUR ET DE MASSE A LA SURFACE DES GOUTTES ET DES BULLES

Résumé—Pour décrire le transfert de chaleur et de masse à l'interface d'un milieu biphasique on élabore un système de critères de similitude basé sur les nombres d'Euler, de Froude, de Galilée, de Jakob, de Nusselt, de Péclet, de Reynolds, de Stanton et de Wéber adaptés pour des gouttes et des bulles. On définit une notion de chaleur latente de capillarité pour rendre compte des déviations causées par l'état de dispersion des particules.

KENNZAHLENGLEICHUNGEN FÜR DEN WÄRME- UND STOFFAUSTAUSCH AN DER OBERFLÄCHE VON TROPFEN UND DAMPFBLASEN

Zusammenfassung—Zur allgemeinen Beschreibung des Wärme- und Stoffaustausches an der Phasengrenzfläche zweiphasiger Stoffe wird ein aus den Kennzahlen *Eu*, *Fr*, *Ga*, *Ja*, *Nu*, *Pe*, *Re*, *St* und *We* bestehender Satz von Ähnlichkeitskriterien erarbeitet. Zur Berücksichtigung der durch den dispersen Zustand der Partikel hervorgerufenen Abweichungen wird der Begriff einer latenten Kapillarwärme definiert.

КРИТЕРИАЛЬНЫЕ УРАВНЕНИЯ ТЕПЛО- И МАССООБМЕНА НА ПОВЕРХНОСТИ КАПЕЛЬ И ПУЗЫРЬКОВ ПАРА

Аннотация — Для описания в общем случае тепло- и массообмена на поверхности раздела двухфазных сред предложена система критериев подобия, включающая числа Эйлера, Фруда, Галилея, Якоба, Нуссельта, Пекле, Рейнольдса, Стантона и Вебера для капель и пузырьков пара. Рассмотрен также учет отклонений, вызванных дисперсным состоянием одной из фаз.